FIXING THE AREA FROM TANGENT LINES OF A CURVE

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Suppose we wish to find a curve $f : (0, \infty) \to (0, \infty)$ such that for any x, the triangle formed by the x-axis, then y-axis, and the tangent line of f at x has area 1. Does one exist, and if so, what are all such curves?

We start by writing out the equation that f needs to satisfy. The slope of the tangent line of f at a is $f'(a)$, so the equation for the tangent line is

$$y - f(a) = f'(a)(x - a)$$

The x-intercept is $a - \frac{f(a)}{f'(a)}$ and the y-intercept is $f(a) - af'(a)$, so the total area is

$$A = 1 = \frac{1}{2}\left(a - \frac{f(a)}{f'(a)}\right)(f(a) - af'(a))$$

After some algebra and using $y = f(x)$ notation, we have

$$-2y' = (xy' - y)^2$$

Taking the derivative of both sides yields

$$-2y'' = 2(xy' - y)(xy'') \tag{1}$$

If $y'' = 0$, then $y = C_1 x + C_2$. In this case, the original differential equation dictates

$$-2C_1 = C_2^2$$

and the general solutions $y = -2C^2 x + 2C$. None of these is defined over $(0, \infty)$, but the collection of solutions is interesting nonetheless. The other case of (1) gives

$$\begin{aligned} -1 &= x^2 y' - xy \\ -x^{-3} &= (x^{-1}y)' \\ \frac{1}{2}x^{-2} + C &= x^{-1}y \\ y &= \frac{1}{2x} + Cx \end{aligned}$$

Plugging this back into the original differential equation to check which C gives a correct solution, we have

$$\begin{aligned} -2y' &= \frac{1}{x^2} - 2C \\ (xy' - y)^2 &= \frac{1}{x^2} \end{aligned}$$

which clearly only works when $C = 0$. Thus the only curve that satisfies the original conditions is $y = \frac{1}{2x}$.

Suppose instead we wanted the triangle to have area $A = g(a) > 0$. Then the differential equation would be

$$-2gy' = (xy' - y)^2$$

The trick of differentiating both sides will not work here since we won't have a y'' to cancel from both sides. Is there a solution in terms of elementary operations? Yes, but we are going to need lots of letters.

First, we will do the substitution $y = xu$ to reduce the squared term to $xy' - y = x^2 u'$.

$$-2gxu' - 2gu = x^4 (u')^2$$

Divide by g and take the derivative to get

$$-2xu'' - 4u' = \left(\frac{x^4 (u')^2}{g}\right)'$$

Date: November 21, 2023.

Letting $v = u'$,

$$-2xv' - 4v = \left(\frac{x^4v^2}{g}\right)'$$

Then we can substitute $v = w\sqrt{g}/x^2$. To make this easier, we can rewrite the left-hand side as $-2xv' - 4v = -\frac{2}{x}\left(x^2v\right)' = -\frac{2}{x}\left(w\sqrt{g}\right)'$

$$\frac{-2w'g - w}{x\sqrt{g}} = 2ww'$$

We can solve for w and take the derivative once again

$$\begin{aligned} w =& \frac{-2gw'}{2x\sqrt{g}w' + 1} \\ w' =& \frac{(2x\sqrt{g}w' + 1)(-2gw'' - 2g'w') + 2gw'(2\sqrt{g}w' + \frac{xw'}{\sqrt{g}} + 2x\sqrt{g}w'')}{(2x\sqrt{g}w' + 1)^2} \\ w'(2x\sqrt{g}w' + 1)^2 =& -2x\sqrt{g}g'(w')^2 - 2gw'' - 2g'w' + 4g\sqrt{g}(w')^2 + 2x\sqrt{g} \end{aligned}$$

If we let $z = w'$, we can solve for z' to get a cubic in z with coefficients arbitrary functions of x.

$$\begin{aligned} z(2x\sqrt{g}z + 1)^2 =& -2x\sqrt{g}g'z^2 - 2gz' - 2g'z + 4g\sqrt{g}z^2 + 2x\sqrt{g} \\ z' =& (-2x^2)z^3 + \left(\frac{2g - x - xg'}{\sqrt{g}}\right)z^2 - \left(\frac{g'}{g}\right)z \end{aligned}$$

This looks reasonable, right? Right? Haha, what innocence. A differential equation of this form has a name, an Abel differential equation of the first kind. The general form for a solution was found in 2011, using the Bessel functions. Let J and Y be the Bessel functions.