CUTTING A SQUARE INTO SMALLER SQUARES

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In math club one day, we did an activity on cutting up a square (WLOG a 1×1 square) exactly into smaller squares, not necessarily different or the same size. The goal was to find the values of $n > 0$ such that it was possible to cut the original square into exactly n smaller squares. This problem is one that I saw on NumberPhile one day, so I won't explain the solution. Instead, I will link to the video at the bottom.

As everybody was working on this problem, one of the members brought up a new question: how many ways could it be done for the possible n's? We called this $f(n)$ and constructed the table

n	1	2	3	5	else
$f(n)$	1	0	0	0	≥ 1

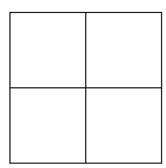
Then we started constructing arguments for $n = 4$ and $n = 6$.

Proposition 0.1. *The value* $f(4) = 1$.

Proof. Each of the 4 small squares must occupy its own corner on the big square. Since the total area is 1, there must be one square with area between $\frac{1}{4}$ and 1, so it has side length $\frac{1}{2} \leq x \leq 1$. The remaining squares must be at most $1 - x$ to also fit in the 1×1 square. The total area is at most

$$x^2 + 3(1-x)^2 = 4x^2 - 6x + 3$$

Setting the area equal to 1, we find $x = \frac{1}{2}$ or $x = 1$. Clearly if $x = 1$, there is no room for the other 3 squares, so $x = \frac{1}{2}$ is the only solution. This gives the expected answer of 1 solution that is



□

Proposition 0.2. *The value* $f(6) = 1$.

Proof. Let the large square be $ABCD$. Like before, we need 4 of the small squares to occupy corners. Suppose these have side lengths a, b, c, d on corners A, B, C, D, respectively. Since we only have 2 additional squares, we can say 2 of the edges must be complete with a, b, c, d. This makes 2 cases: the completed edges are opposite each other or adjacent. We cover the opposite case first.

WLOG, let $a + b = c + d = 1$ to complete the edges AB and CD. Then one of a, b is at least $\frac{1}{2}$ and one of c, d is at least $\frac{1}{2}$. However, these squares overlap unless $a = b = c = d = \frac{1}{2}$, but then there is no room for the remaining 2 squares. Thus the completed edges must be adjacent.

WLOG, let $a + b = a + d = 1$ to complete the edges AB and AD. Like before, either $a \geq \frac{1}{2}$ or $b = d > \frac{1}{2}$. In the second case, the squares overlap, so we have $a \geq \frac{1}{2}$. The remaining squares have side lengths $1 - c - b$ and $1 - c - d$, but since $b = d = 1 - a$, these values are both $x = a - c$. Note that in order to fit, we have $c \leq 1 - a$ and $x = a - c \leq 1 - a$. We rearrange these to get $2a - 1 \leq c \leq 1 - a$, so $a \leq \frac{2}{3}$. Also, if $x > c$, then the last 2 squares overlap, so $x = a - c \leq c$, i.e., $c \geq \frac{1}{2}a$. The total area is

$$a^2 + 2(1-a)^2 + c^2 + 2(a-c)^2$$

We claim under our conditions, the maximum area is 1 and we solve for this maximum. This equation for area has a maximum in c at either boundary point $c = \frac{1}{2}a$ or $c = 1 - a$. In the first case, we use the boundary points $a = \frac{1}{2}$ and $a = \frac{2}{3}$ to get the maximum area is 1 at $a = \frac{2}{3}$, $c = \frac{1}{3}$, corresponding to the solution

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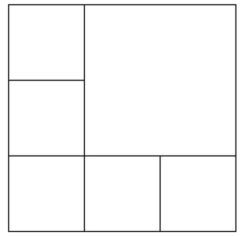
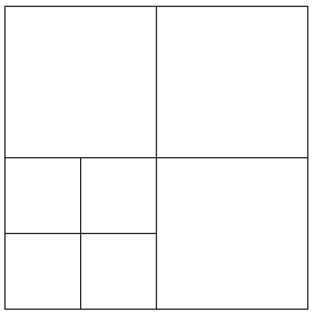
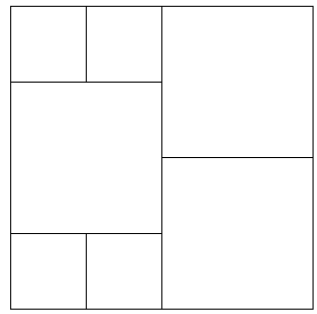


In the other case, we use the same method to get the maximum area 1 as a repeat solution and $a = c = \frac{1}{2}$. In the second case, however, $x = 0$ and the solution is degenerate. Thus $f(6) = 1$. □

One might wonder if the values of f are always 0 or 1, but this is quickly disproven, as $f(7) \geq 2$ shown by

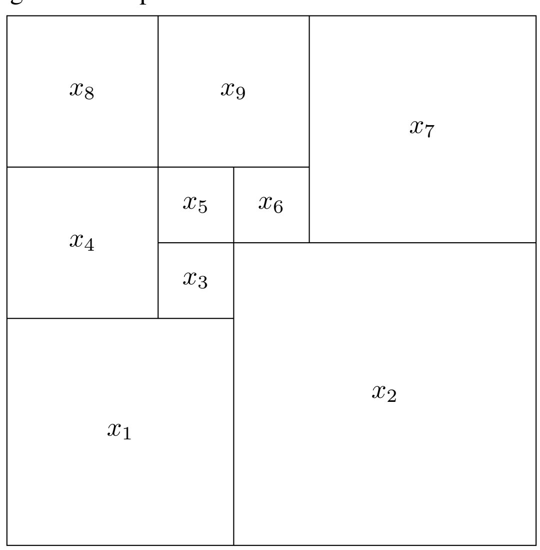




What is $f(7)$? I don't know. But this seems like a good time to update our table:

n	1	2	3	4	5	6	7	else
$f(n)$	1	0	0	1	0	1	≥ 2	≥ 1

With more squares, the arguments are only going to get longer and we won't be able to argue about $n \to \infty$, so let's look for a different setup for the problem. Suppose we are given an arrangement of squares and we are tasked with finding the sizes of the small squares. Let's work through the example



where the x_n represents the side length. We start analyzing the possible horizontal lines through this square starting at the bottom. We have the equation

$$x_1 + x_2 = 1$$

Once our horizontal line reaches the end of square x_1, we get a new equation

$$x_4 + x_3 + x_2 = 1$$

We can see that doing this will give us one new equation every time we reach the end of a square, almost. We could run into an issue with the end of square x_2 because it is the same as the square x_4. However, this fact gives us enough information to construct a new equation: since the squares ended at the same location, we have the equation from the vertical positions:

$$x_3 + x_1 = x_2$$

We continue this pattern to get the rest of the equations:

$$\begin{aligned} x_4+x_5+x_6+x_7 =& 1 \\ x_4+x_1 =& x_5+x_3+x_1 \\ x_5+x_3+x_1 =& x_6+x_2 \\ x_8+x_9+x_7 =& 1 \\ x_8+x_4+x_1 =& x_9+x_5+x_3+x_1 \\ x_9+x_5+x_3+x_1 =& x_7+x_2 \end{aligned}$$

and reach the top of the square. Putting these linear equations in matrix form, we have

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Letting A be the square matrix, X the column vector of x_n's, and B the right column vector, we solve and find

$$X = A^{-1}B = (3/7, 4/7, 1/7, 2/7, 1/7, 1/7, 3/7, 2/7, 2/7)^T$$

The horizontal sums and vertical sums must be linearly independent, so A will always be invertible. We can also do better with restricting A. Take a row with -1 such as $x_3 + x_1 = x_2$. This came from equating vertical lines. If we extend one vertical line, then we have

$$x_1 + x_3 + x_5 + x_9 = 1 = x_2 + x_5 + x_9$$

One of the equalities to 1 is linearly independent from the rest of the rows of A, so we can rewrite A using the new equation. This produces a system $AX = B$ where A is a $\{0,1\}$ matrix and B is the column vector of 1's. Solving $X = A^{-1}B$ gives a finite, albeit large number of possibilities for X. We can extend this to a nice proposition:

Proposition 0.3. *For any n, $f(n)$ is finite. In particular, it is bounded by $2^{n^2}n!$*

Proof. Any geometric construction has a system that arises as in the previous paragraphs. There are at most 2^{n^2} possible matrices for A. Choosing a general ordering to placing the squares geometrically (such as sorted by x-coordinate, then y-coordinate) gives at most $2^{n^2}n!$ geometric arrangements. ☐

The upper bound provided in the proposition leaves much to be desired. At $n = 6$, for example, $2^{6^2}6! = 49478023249920$, which is a bit more than the proven value $f(6) = 2$.

We can try getting as much information as possible from this system. Since

$$\det(A)\, A^{-1}$$

is an integer matrix, so must

$$\det(A)\, X = \left(\det(A)\, A^{-1}\right) B$$

so we can deduce that the maximum possible denominator for a value in X is $\det(A)$. What is a bound on $\det(A)$, then? Results on an almost identical problem, Hadamard's Maximal Determinant Problem, and a connection to this one show

$$\det(A) \leq \frac{(n+1)^{(n+1)/2}}{2^n}$$

Before we put up our hat because this is current research by the top mathematicians, we can make one more appeal to geometry. We still have not used the facts that

$$|X|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = 1$$

and each $x_i > 0$. We put these equations into a single statement.

Problem 0.1. Suppose the following for matrices A, X, B

- $A \in \{0,1\}^{n\times n}$
- $X = (x_1, \cdots, x_n)^T$ with each $x_i > 0$

- $B = (1, \cdots, 1)^T$
- $|X| = 1$

What are the possible values for $\det(A)$?