FINDING UNIQUE SUM-PRODUCT-LENGTH OF DIGITS

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What numbers are uniquely determined by the number of digits, the sum of digits, and the product of digits? For example, what 2-digit number has product of digits 4 and sum of digits 4? Clearly the only answer is 22. What 2-digit number has product of digits 35 and sum of digits 8? Now, there are two possibly answers: 35 and 53. Here, we have the option to rephrase the question. Do we care about the order of the digits? For now, we will care about the order, so the 35 ad 53 are considered different. We will call the unique numbers like 22 savvy in base 10. We extend this question to any base b. There are two cases to consider: when a number contains a digit 0 and when it does not.

Denote the sum of the digits as $S(N)$ and the product of the digits as $P(N)$, in base b. We first consider the case that N is a savvy number and it contains a digit 0. This is exactly the case when the $P(N) = 0$. Clearly, the only 1-digit savvy number in this case is 0. For 2-digit numbers, the units digit must be the 0. The sum then determines the tens digit. If N has 3 or more digits, then there is a choice where in the number to put the 0, unless every digit but the leading digit is 0. If the sum is 1, then $N = 100\cdots 0$. Otherwise, it is possible to split $S(N)$ into multiple digits, for example $S(N) = 2$ could have solutions $N = 1100\cdots 0$ or $1010\cdots 0$. Thus the savvy numbers N for which $P(N) = 0$ are

$$0, 10, 20, \cdots, 100, 1000, \cdots$$

The only question left, then, is whether every repeated digit number is the only number for its set of length, product, and sum. For example, is 66666 is the only 3-digit number with sum $30 = 5 \cdot 6$ and product $7776 = 6^5$? Surprisingly, this is very simple to figure out. The AM-GM inequality states that the arithmetic mean of a list of numbers is always at least the geometric mean, with equality exactly when the list is a single repeated number. If we have a digit d repeated n times, then the arithmetic mean and geometric mean are both d. Thus the number must be $ddd\cdots$. Thus in any base, the savvy numbers are the multiples of 10 up to 100, the powers of 10, and the repdigits.

Let's loosen the requirement on caring about the order of the digits, so 35 and 53 are considered equivalent.

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